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储层流固耦合的数学模型和非线性有限元方程

张广明¹, 刘 合^{1,2}, 张 劲³, 吴恒安¹, 王秀喜¹

(1.中国科学技术大学 近代力学系, 合肥 230026; 2.中国石油勘探开发研究院, 北京 100083;
3.中国石油大学(北京) 石油和天然气工程学院, 北京 102200)

摘 要: 根据饱和多孔介质固体骨架的平衡方程和多孔介质中流体的连续性方程, 建立了储层流固耦合数学模型。模型中引入了 Jaumann 应力速率公式描述多孔介质固体骨架的大变形效应, 并考虑了地应力、初始孔隙压力、初始流体密度和初始孔隙度对耦合模型的影响。基于与微分方程等价的加权余量公式, 在空间域采用有限元离散, 对时间域进行隐式差分格式离散, 导出了以单元节点位移和单元节点孔隙压力为未知量的储层流固耦合的非线性有限元增量方程。该模型在石油工程中有广泛的应用, 为储层流固耦合的数值模拟奠定了理论基础。

关 键 词: 储层; 流固耦合; 数学模型; 非线性有限元; 多孔介质

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Mathematical model and nonlinear finite element equation for reservoir fluid-solid coupling

ZHANG Guang-ming¹, LIU He^{1,2}, ZHANG Jin³, WU Heng-an¹, WANG Xiu-xi¹

(1. Department of Modern Mechanics, Hefei 230026, China; 2. Research Institute of Petroleum Exploration Development, Petro China, Beijing 163453, China;
3. School of Petroleum and Gas Engineering, China University of Petroleum, Beijing 102200, China)

Abstract: A mathematical model for fluid-solid coupling in reservoir was constructed based on the equilibrium equation of solid skeleton and continuity equation of wetting liquid in porous medium. Jaumann stress rate formula was adopted to describe the finite deformation behavior. The effects such as in situ stress, initial pore pressure, initial fluid density and initial porosity of the medium was also taken into account in the mathematical model. Finite element discretization was carried out on the mathematical model to get a fully coupled finite element equation with the parameters of element nodal displacement and element nodal pore pressure as unknown variables. Finally, the transient nonlinear finite element equation in incremental form for reservoir fluid-solid coupling is obtained by means of time discretization in implicit difference scheme. The resulted finite element formulation can simulate the reservoir behavior comprehensively and would have wide applications in petroleum engineering.

Key words: reservoir; fluid-solid coupling; mathematical model; nonlinear finite elements; porous medium

1 引 言

Terzaghi^[1]最早研究了可变形饱和多孔介质中的流体流动, 提出了有效应力原理, 建立了一维固结理论。Biot^[2-5]利用线弹性本构关系和 Darcy 定律, 建立了比较完善的三维固结理论。董平川和徐小荷^[6]基于小变形理论, 建立了储层流固耦合的有限元方程。李培超等^[7]基于多孔介质有效应力原理, 建立了饱和多孔介质流固耦合渗流的数学模型。程涛等^[8]对耦合方程的解耦求法进行了研究。韩炜洁等^[9]对状态方程在流固耦合方程解法中的应用进行

了探讨。本文基于大变形理论假设, 根据多孔介质固体骨架的平衡方程和流体的连续性方程, 建立了储层流固耦合数学模型, 并建立了以单元节点位移和单元节点孔隙压力为未知量的完全耦合的瞬态非线性增量有限元方程。

2 储层流固耦合数学模型

2.1 多孔介质的控制方程

多孔介质固体骨架的平衡方程^[10]为

$$\sigma_{ij,j} + f_i = 0, \text{ (在区域 } V \text{ 内)} \quad (1)$$

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第一作者简介: 张广明, 男, 1980年生, 博士研究生, 主要从事石油工程数值模拟研究。E-mail: zhgm@mail.ustc.edu.cn

边界条件为

$$\left. \begin{aligned} \sigma_{ij}n_j - t_i &= 0, \quad (\text{在力的边界 } S_\sigma \text{ 上}) \\ u_i &= \bar{u}_i, \quad (\text{在位移边界 } S_u \text{ 上}) \end{aligned} \right\} \quad (2)$$

式中: σ_{ij} 为应力分量 (Pa); $\sigma_{ij,j}$ 为 σ_{ij} 对坐标 x_j 的偏导数 (N/m³); f_i 为体积力分量 (N/m³); n_j 为边界的外法向分量, 无因次; t_i 为表面力分量 (Pa); u_i 为位移分量 (m); $\bar{u}_i$ 为边界上指定的位移分量 (m)。

平衡方程的权函数取真实速度的变分 δv_i , 力边界条件的权函数取 $-\delta v_i$, 得到经过分部积分后“弱”形式的矩阵表示为

$$\int_V \delta \dot{\boldsymbol{\varepsilon}}^T \boldsymbol{\sigma} dV = \int_{S_\sigma} \delta \mathbf{v}^T \mathbf{t} dS + \int_V \delta \mathbf{v}^T \mathbf{f} dV \quad (3)$$

式中: $\boldsymbol{\sigma}$ 为应力矩阵 (Pa); $\delta \dot{\boldsymbol{\varepsilon}}$ 为虚应变率矩阵 (1/s); $\mathbf{t}$ 为表面力向量 (Pa); $\mathbf{f}$ 为体积力向量 (N/m³); $\delta \mathbf{v}$ 为虚速度向量 (m/s)。

在应力渗流耦合分析中使用有效应力原理^[11], 有效应力的矩阵表达式为

$$\bar{\boldsymbol{\sigma}} = \boldsymbol{\sigma} + p_w \mathbf{m} = \mathbf{D} \left(\boldsymbol{\varepsilon} + \frac{p_w - p_w^0}{3K_g} \mathbf{m} \right) + \bar{\boldsymbol{\sigma}}_0 \quad (4)$$

式中: $\bar{\boldsymbol{\sigma}}$ 为有效应力矩阵 (Pa); p_w 为流体孔隙压力 (Pa); $\mathbf{m} = [1, 1, 1, 0, 0, 0]^T$; $\mathbf{D}$ 为弹塑性矩阵 (Pa); K_g 为多孔介质固体骨架体积模量 (Pa); p_w^0 为流体初始孔隙压力 (Pa); $\bar{\boldsymbol{\sigma}}_0$ 为地应力矩阵 (Pa)。

从式 (4) 可以看出, 流体孔隙压力对固体骨架有效应力的贡献, 有效应力以拉应力为正, 流体孔隙压力以压应力为正。

对式 (3) 进行有限元离散可以得到

$$\int_V \mathbf{B}^T \boldsymbol{\sigma} dV = \int_{S_\sigma} \mathbf{N}^T \mathbf{t} dS + \int_V \mathbf{N}^T \mathbf{f} dV \quad (5)$$

式中: $\mathbf{B}$ 为单元应变矩阵; $\mathbf{N}$ 为单元位移插值函数矩阵。

对式 (5) 等号两端求微分, 考虑大变形效应并认为表面力和体力矢量为保守力载荷^[12], 可得

$$\int_V \left[\frac{1}{J} \mathbf{B}^T d(\mathbf{J}\boldsymbol{\sigma}) + (d\mathbf{B}^T) \boldsymbol{\sigma} \right] dV = \int_{S_\sigma} \mathbf{N}^T d\mathbf{t} dS + \int_V \mathbf{N}^T d\mathbf{f} dV \quad (6)$$

把单元应变矩阵 $\mathbf{B}$ 写成线性应变矩阵 $\mathbf{B}_L$ 和非线性应变矩阵 $\mathbf{B}_N$ 之和, 则有

$$\begin{aligned} (d\mathbf{B}^T) \boldsymbol{\sigma} &= (d\mathbf{B}_L^T + d\mathbf{B}_N^T) \boldsymbol{\sigma} = \\ (d\mathbf{B}_N^T) \boldsymbol{\sigma} &= [d(\mathbf{G}^T \mathbf{A}^T)] \boldsymbol{\sigma} = \mathbf{G}^T (d\mathbf{A}^T) \boldsymbol{\sigma} = \end{aligned}$$

$$\mathbf{G}^T \begin{bmatrix} \sigma_{11} \frac{\partial \mathbf{u}}{\partial x_1} + \sigma_{12} \frac{\partial \mathbf{u}}{\partial x_2} + \sigma_{31} \frac{\partial \mathbf{u}}{\partial x_3} \\ \sigma_{12} \frac{\partial \mathbf{u}}{\partial x_1} + \sigma_{22} \frac{\partial \mathbf{u}}{\partial x_2} + \sigma_{23} \frac{\partial \mathbf{u}}{\partial x_3} \\ \sigma_{31} \frac{\partial \mathbf{u}}{\partial x_1} + \sigma_{23} \frac{\partial \mathbf{u}}{\partial x_2} + \sigma_{33} \frac{\partial \mathbf{u}}{\partial x_3} \end{bmatrix} = \mathbf{G}^T \begin{bmatrix} (\sigma_{11} \frac{\partial}{\partial x_1} + \sigma_{12} \frac{\partial}{\partial x_2} + \sigma_{31} \frac{\partial}{\partial x_3}) \mathbf{I} \\ (\sigma_{12} \frac{\partial}{\partial x_1} + \sigma_{22} \frac{\partial}{\partial x_2} + \sigma_{23} \frac{\partial}{\partial x_3}) \mathbf{I} \\ (\sigma_{31} \frac{\partial}{\partial x_1} + \sigma_{23} \frac{\partial}{\partial x_2} + \sigma_{33} \frac{\partial}{\partial x_3}) \mathbf{I} \end{bmatrix} d\mathbf{u} = \mathbf{G}^T \mathbf{S} d\mathbf{u} \quad (7)$$

式中: $\mathbf{B}_L$ 、 $\mathbf{B}_N$ 、 $\mathbf{G}$ 、 $\mathbf{A}$ 的表达式见文献[12]。

在大变形问题中, 采用的度量是 Kirchhoff 应力 $\mathbf{J}\boldsymbol{\sigma}$ 和 Green 应变 $\boldsymbol{\varepsilon}$ ^[12], 又有

$$J = \frac{dV}{dV^0} = \left| \frac{\partial \mathbf{x}}{\partial \mathbf{X}} \right| \quad (8)$$

式中: $\mathbf{X}$ 和 $\mathbf{x}$ 分别为空间点初始位置矢量和现时位置矢量。应力和应变矢量分别记为

$$\boldsymbol{\sigma} = [\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{23}, \sigma_{31}, \sigma_{12}]^T \quad (9)$$

$$\boldsymbol{\varepsilon} = [\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, 2\varepsilon_{23}, 2\varepsilon_{31}, 2\varepsilon_{12}]^T \quad (10)$$

为表征大变形情况下应力变化随刚体转动的坐标不变性, 引入如下共转应力速率 (Jaumann 应力速率) 公式^[13]:

$$d^\nabla(\mathbf{J}\boldsymbol{\sigma}) = d(\mathbf{J}\boldsymbol{\sigma}) - J(d\boldsymbol{\Omega}\boldsymbol{\sigma} + \boldsymbol{\sigma}d\boldsymbol{\Omega}^T) \quad (11)$$

根据有效应力原理:

$$\begin{aligned} d^\nabla(\mathbf{J}\boldsymbol{\sigma}) &= d^\nabla(\mathbf{J}\bar{\boldsymbol{\sigma}}) - d(Jp_w)\mathbf{m} = \\ d^\nabla(\mathbf{J}\bar{\boldsymbol{\sigma}}) - p_w J \mathbf{m} \mathbf{m}^T d\boldsymbol{\varepsilon} - J dp_w \mathbf{m} \end{aligned} \quad (12)$$

式中: $d^\nabla(\mathbf{J}\bar{\boldsymbol{\sigma}})$ 为不随材料刚体旋转而变化的客观张量, 只与材料的本构有关, 即

$$d^\nabla(\mathbf{J}\bar{\boldsymbol{\sigma}}) = J \mathbf{D} d\boldsymbol{\varepsilon} + \frac{J \mathbf{D} \mathbf{m}}{3K_g} dp_w \quad (13)$$

由式 (11)~(13), 可得

$$\begin{aligned} d(\mathbf{J}\boldsymbol{\sigma}) &= J \mathbf{D} d\boldsymbol{\varepsilon} + \frac{J \mathbf{D} \mathbf{m}}{3K_g} dp_w - p_w J \mathbf{m} \mathbf{m}^T d\boldsymbol{\varepsilon} - \\ &J dp_w \mathbf{m} + J(d\boldsymbol{\Omega}\boldsymbol{\sigma} + \boldsymbol{\sigma}d\boldsymbol{\Omega}^T) \end{aligned} \quad (14)$$

式中: $d\boldsymbol{\Omega} = \text{asym} \left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}} \right)$ 为材料的旋转增量, 无因

次, 这里 $d\Omega$ 和 σ 为 3×3 的矩阵, σ 为 6×1 的矢量, 所以 $J(d\Omega\sigma + \sigma d\Omega^T)$ 和 $d(J\sigma)$ 不能进行矩阵相

加, 有限元列式中 σ 为 6×1 的矢量, 所以要把 3×3 的矩阵 $d\Omega\sigma + \sigma d\Omega^T$ 转换为 6×1 的矢量。记

$$d\Omega = \begin{bmatrix} 0 & \Omega_1 & \Omega_2 \\ -\Omega_1 & 0 & \Omega_3 \\ -\Omega_2 & -\Omega_3 & 0 \end{bmatrix}, \quad \sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{31} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{23} & \sigma_{33} \end{bmatrix} \quad (15)$$

$$\text{则 } d\Omega\sigma + \sigma d\Omega^T = \begin{bmatrix} 2\Omega_1\sigma_{12} + 2\Omega_2\sigma_{31} & -\Omega_1\sigma_{11} + \Omega_1\sigma_{22} + \Omega_2\sigma_{23} + \Omega_3\sigma_{31} & -\Omega_2\sigma_{11} + \Omega_2\sigma_{33} + \Omega_1\sigma_{23} - \Omega_3\sigma_{12} \\ \text{对} & -2\Omega_1\sigma_{12} + 2\Omega_3\sigma_{23} & -\Omega_3\sigma_{22} + \Omega_3\sigma_{33} - \Omega_1\sigma_{13} - \Omega_2\sigma_{12} \\ \text{称} & & -2\Omega_2\sigma_{13} - 2\Omega_3\sigma_{23} \end{bmatrix} \quad (16)$$

把 3×3 的矩阵 $d\Omega\sigma + \sigma d\Omega^T$ 按照 Kirchhoff 应力格式写成 6×1 的矢量, 有

$$\begin{bmatrix} 2\Omega_1\sigma_{12} + 2\Omega_2\sigma_{31} \\ -2\Omega_1\sigma_{12} + 2\Omega_3\sigma_{23} \\ -2\Omega_2\sigma_{13} - 2\Omega_3\sigma_{23} \\ -\Omega_3\sigma_{22} + \Omega_3\sigma_{33} - \Omega_1\sigma_{13} - \Omega_2\sigma_{12} \\ -\Omega_2\sigma_{11} + \Omega_2\sigma_{33} + \Omega_1\sigma_{23} - \Omega_3\sigma_{12} \\ -\Omega_1\sigma_{11} + \Omega_1\sigma_{22} + \Omega_2\sigma_{23} + \Omega_3\sigma_{31} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 2\Omega_2 & 2\Omega_1 \\ 0 & 0 & 0 & 2\Omega_3 & 0 & -2\Omega_1 \\ 0 & 0 & 0 & -2\Omega_3 & -2\Omega_1 & 0 \\ 0 & -\Omega_3 & \Omega_3 & 0 & -\Omega_1 & -\Omega_2 \\ -\Omega_2 & 0 & \Omega_2 & \Omega_1 & 0 & -\Omega_3 \\ -\Omega_1 & \Omega_1 & 0 & \Omega_2 & \Omega_3 & 0 \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{bmatrix} = \tilde{T}\sigma \quad (17)$$

又由

$$\Omega_1 = \frac{1}{2} \left(\frac{\partial du_1}{\partial x_2} - \frac{\partial du_2}{\partial x_1} \right), \quad \Omega_2 = \frac{1}{2} \left(\frac{\partial du_1}{\partial x_3} - \frac{\partial du_3}{\partial x_1} \right), \quad \Omega_3 = \frac{1}{2} \left(\frac{\partial du_2}{\partial x_3} - \frac{\partial du_3}{\partial x_2} \right) \quad (18)$$

记

$$\left. \begin{aligned} T_1 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{1}{2} \frac{\partial}{\partial x_3} & 0 & \frac{1}{2} \frac{\partial}{\partial x_1} \\ -\frac{1}{2} \frac{\partial}{\partial x_2} & \frac{1}{2} \frac{\partial}{\partial x_1} & 0 \end{bmatrix}, \quad T_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -\frac{1}{2} \frac{\partial}{\partial x_3} & \frac{1}{2} \frac{\partial}{\partial x_2} \\ \frac{1}{2} \frac{\partial}{\partial x_2} & -\frac{1}{2} \frac{\partial}{\partial x_1} & 0 \end{bmatrix}, \quad T_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{1}{2} \frac{\partial}{\partial x_3} & -\frac{1}{2} \frac{\partial}{\partial x_2} \\ \frac{1}{2} \frac{\partial}{\partial x_3} & 0 & -\frac{1}{2} \frac{\partial}{\partial x_1} \\ 0 & 0 & 0 \end{bmatrix}, \\ T_4 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{\partial}{\partial x_3} & -\frac{\partial}{\partial x_2} \\ 0 & -\frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_2} \\ 0 & 0 & 0 \\ \frac{1}{2} \frac{\partial}{\partial x_2} & -\frac{1}{2} \frac{\partial}{\partial x_1} & 0 \\ \frac{1}{2} \frac{\partial}{\partial x_3} & 0 & -\frac{1}{2} \frac{\partial}{\partial x_1} \end{bmatrix}, \quad T_5 = \begin{bmatrix} \frac{\partial}{\partial x_3} & 0 & -\frac{\partial}{\partial x_1} \\ 0 & 0 & 0 \\ -\frac{\partial}{\partial x_3} & 0 & \frac{\partial}{\partial x_1} \\ -\frac{1}{2} \frac{\partial}{\partial x_2} & \frac{1}{2} \frac{\partial}{\partial x_1} & 0 \\ 0 & 0 & 0 \\ 0 & \frac{1}{2} \frac{\partial}{\partial x_3} & -\frac{1}{2} \frac{\partial}{\partial x_2} \end{bmatrix}, \quad T_6 = \begin{bmatrix} \frac{\partial}{\partial x_2} & -\frac{\partial}{\partial x_1} & 0 \\ -\frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} & 0 \\ 0 & 0 & 0 \\ -\frac{1}{2} \frac{\partial}{\partial x_3} & 0 & \frac{1}{2} \frac{\partial}{\partial x_1} \\ 0 & -\frac{1}{2} \frac{\partial}{\partial x_3} & \frac{1}{2} \frac{\partial}{\partial x_2} \\ 0 & 0 & 0 \end{bmatrix} \end{aligned} \right\} \quad (19)$$

$$\left. \begin{aligned} I_1 &= [1, 0, 0, 0, 0, 0]; \quad I_2 = [0, 1, 0, 0, 0, 0] \\ I_3 &= [0, 0, 1, 0, 0, 0]; \quad I_4 = [0, 0, 0, 1, 0, 0] \\ I_5 &= [0, 0, 0, 0, 1, 0]; \quad I_6 = [0, 0, 0, 0, 0, 1] \end{aligned} \right\} \quad (20)$$

可以得到式 (21)。

$$\begin{aligned} \tilde{T}\sigma &= (T_1 du I_1 + T_2 du I_2 + T_3 du I_3 + T_4 du I_4 + T_5 du I_5 + \\ &\quad T_6 du I_6) \sigma = \sigma_{11} T_1 du + \sigma_{22} T_2 du + \sigma_{33} T_3 du + \\ &\quad \sigma_{23} T_4 du + \sigma_{31} T_5 du + \sigma_{12} T_6 du = (\sigma_{11} T_1 + \sigma_{22} T_2 + \\ &\quad \sigma_{33} T_3 + \sigma_{23} T_4 + \sigma_{31} T_5 + \sigma_{12} T_6) du = T du \end{aligned} \quad (21)$$

把式 (21) 代入式 (14), 可得

$$d(J\sigma) = JDd\epsilon + \frac{JDm}{3K_g} dp_w - p_w Jmm^T d\epsilon - Jmdp_w + JTdu \quad (22)$$

把式 (7)、(22) 代入式 (6), 可得

$$\int_V [B^T Dd\epsilon + \frac{B^T Dm}{3K_g} dp_w - B^T p_w mm^T d\epsilon - B^T mdp_w + B^T Tdu + G^T Sdu] dV = \int_{S_q} N^T dt dS + \int_V N^T df dV \quad (23)$$

2.2 流体连续性方程

流体连续性方程的微分形式^[13]为

$$\frac{1}{J} \frac{\partial}{\partial t} (J\rho_w n_w) + \frac{\partial}{\partial x} (\rho_w n_w v_w) = 0 \quad (24)$$

式中: ρ_w 为流体密度 (kg/m^3), $\rho_w = \rho_w^0 \left(1 + \frac{p_w}{K_w}\right)$;

ρ_w^0 为流体初始时刻密度 (kg/m^3); K_w 为流体体积模量 (Pa); n_w 为流体体积与总体积的比率, 无因次; v_w 为流体渗流速度 (m/s)。

式 (24) 包含了系统体积变化对流体连续性方程的影响, 考虑了流固耦合效应。

对于饱和的多孔介质,

$$n_w = 1 - \frac{m^T \bar{\sigma}}{3K_g} + \frac{1}{J} (1 - n^0) \left(\frac{p_w}{K_g} - 1 \right) \quad (25)$$

式中: n^0 为多孔介质初始孔隙度, 无因次。

式 (25) 同时体现了固体骨架有效应力和流体孔隙压力对流体体积的影响, 同样考虑了流固耦合效应。

多孔介质流体流动服从达西定律, 其方程为

$$v_w = -\frac{1}{n_w g \rho_w} k \left(\frac{\partial p_w}{\partial x} - \rho_w g \right) \quad (26)$$

式中: k 为多孔介质有效渗透率向量 (m/s); g 为重力加速度向量 (m/s^2)。

把式 (25)、(26) 代入式 (24), 可得

$$E \equiv \rho_w^0 \left(1 + \frac{p_w}{K_w} \right) \left[Jm^T - \frac{Jm^T \bar{\sigma} m^T}{3K_g} - \frac{Jm^T D}{3K_g} \right] \frac{\partial \epsilon}{\partial t} + \left[\frac{\rho_w^0 J}{K_w} - \frac{J\rho_w^0 m^T \bar{\sigma}}{3K_g K_w} + \frac{\rho_w^0}{K_w} (1 - n^0) \left(\frac{p_w}{K_g} - 1 \right) - \frac{Jm^T Dm}{9K_g^2} + \frac{1 - n^0}{K_g} \right] \frac{\partial p_w}{\partial t} - \frac{Jk}{g} \frac{\partial^2 p_w}{\partial x^2} + \frac{\rho_w^0 Jkg}{gK_w} \frac{\partial p_w}{\partial x} = 0 \quad (27)$$

流体连续性方程的边界条件为

$$F = \begin{cases} -\frac{n^T}{n_w g \rho_w} k \left(\frac{\partial p_w}{\partial x} - \rho_w g \right) - \bar{q} = 0, & \text{(在流体载荷边界 } S_q \text{ 上)} \\ p_w - \bar{p}_w = 0, & \text{(在孔隙压力边界 } S_p \text{ 上)} \end{cases} \quad (28)$$

式中: $\bar{q}$ 为给定的单位面积的流体体积流量向量 (m/s); $\bar{p}_w$ 为指定的孔隙压力 (Pa)。

将连续性方程式 (27) 应用到整个区域 V 和将边界条件式 (28) 应用到流体载荷边界 S_q 上, 得到微分方程的等效积分形式为

$$\int_V a E dV + \int_{S_q} b F dS = 0 \quad (29)$$

式中: a 和 b 为任意函数。

根据格林定理消去 p_w 的 2 阶偏导数, 取 $b = -aJ\rho_w n_w$, 把式 (4) 代入式 (27), 并注意在孔隙压力边界上, $\frac{\partial p_w}{\partial x} = 0$, 由式 (29) 整理得

$$\begin{aligned} & \int_V a \left\{ \rho_w^0 \left(1 + \frac{p_w}{K_w} \right) \left[Jm^T - \frac{Jm^T D \epsilon m^T}{3K_g} - \frac{Jm^T Dm (p_w - p_w^0) m^T}{9K_g^2} - \frac{Jm^T \bar{\sigma}^0 m^T}{3K_g} - \frac{Jm^T D}{3K_g} \right] \frac{\partial \epsilon}{\partial t} + \left[\frac{\rho_w^0 J}{K_w} - \frac{\rho_w^0 J}{K_w} \left(\frac{m^T D \epsilon}{3K_g} + \frac{m^T Dm (p_w - p_w^0)}{9K_g^2} + \frac{m^T \bar{\sigma}^0}{3K_g} \right) + \frac{\rho_w^0}{K_w} (1 - n^0) \left(\frac{p_w}{K_g} - 1 \right) - \frac{Jm^T Dm}{9K_g^2} + \frac{1 - n^0}{K_g} \right] \frac{\partial p_w}{\partial t} \right\} dV + \\ & \int_V a \left(\frac{Jk}{g} \frac{\partial a}{\partial x} \frac{\partial p_w}{\partial x} + \frac{\rho_w^0 Jkg}{gK_w} \frac{\partial p_w}{\partial x} \right) dV - \\ & \int_{S_q} a J \left\{ \frac{n^T kg}{g} - \rho_w^0 \left(1 + \frac{p_w}{K_w} \right) \left[\bar{q} - \frac{\bar{q} m^T D \epsilon}{3K_g} - \frac{\bar{q} m^T D (p_w - p_w^0) m}{9K_g^2} - \frac{\bar{q} m^T \bar{\sigma}}{3K_g} + \frac{\bar{q} (1 - n^0)}{J} \right] \right\} dS = 0 \end{aligned} \quad (30)$$

3 系统有限元方程

对式 (23) 进行有限元离散可得

$$\begin{aligned} & \int_V (B^T DB - B^T N_p p_w^e mm^T B + B^T TN + G^T SN) \cdot \\ & dV du^e + \int_V \left(\frac{B^T Dm}{3K_g} - B^T m \right) N_p dV dp_w^e = \\ & \int_{S_q} N^T dt dS + \int_V N^T df dV \end{aligned} \quad (31)$$

式中： $\mathbf{u}^e$ 为单元的节点位移矢量（m）； $\mathbf{N}_p$ 为单元节点孔隙压力插值函数； $\mathbf{p}_w^e$ 为单元的节点孔隙压力（Pa）。

采用 Galerkin 法，取 $\mathbf{a} = \mathbf{N}_p$ ，对式（30）进行有限元离散，整理得到

$$\begin{aligned} & \int_V \mathbf{N}_p \rho_w^0 \left(1 + \frac{\mathbf{N}_p \mathbf{p}_w^e}{K_w} \right) \left[\mathbf{J} \mathbf{m}^T \mathbf{B} - \frac{\mathbf{J} \mathbf{m}^T \mathbf{D} \tilde{\mathbf{B}} \mathbf{u}^e \mathbf{m}^T \mathbf{B}}{3K_g} - \right. \\ & \left. \frac{\mathbf{J} \mathbf{m}^T \mathbf{D} \mathbf{m} (\mathbf{N}_p \mathbf{p}_w^e - \rho_w^0) \mathbf{m}^T \mathbf{B}}{9K_g^2} - \frac{\mathbf{J} \mathbf{m}^T \tilde{\boldsymbol{\sigma}}^0 \mathbf{m}^T \mathbf{B}}{3K_g} - \frac{\mathbf{J} \mathbf{m}^T \mathbf{D} \mathbf{B}}{3K_g} \right] \\ & dV \frac{d\mathbf{u}^e}{dt} + \int_V \mathbf{N}_p \left[\frac{\rho_w^0 J}{K_w} - \frac{\rho_w^0 J}{K_w} \left(\frac{\mathbf{m}^T \mathbf{D} \tilde{\mathbf{B}} \mathbf{u}^e}{3K_g} + \right. \right. \\ & \left. \left. \frac{\mathbf{m}^T \mathbf{D} \mathbf{m} (\mathbf{N}_p \mathbf{p}_w^e - \rho_w^0)}{9K_g^2} + \frac{\mathbf{m}^T \tilde{\boldsymbol{\sigma}}^0}{3K_g} \right) + \frac{\rho_w^0}{K_w} (1 - n^0) \cdot \right. \\ & \left. \left(\frac{\mathbf{N}_p \mathbf{p}_w^e}{K_g} - 1 \right) - \frac{\mathbf{J} \mathbf{m}^T \mathbf{D} \mathbf{m}}{9K_g^2} + \frac{1 - n^0}{K_g} \right] \mathbf{N}_p dV \frac{d\mathbf{p}_w^e}{dt} + \\ & \int_V \mathbf{N}_p \left(\frac{Jk}{g} \frac{\partial \mathbf{N}_p}{\partial \mathbf{x}} \frac{\partial \mathbf{N}_p}{\partial \mathbf{x}} \mathbf{p}_w^e + \frac{\rho_w^0 Jk}{gK_w} \frac{\partial \mathbf{N}_p}{\partial \mathbf{x}} \mathbf{p}_w^e \right) dV - \\ & \int_{S_q} \frac{\mathbf{N}_p J \rho_w^0 \bar{\mathbf{q}} \mathbf{m}^T \mathbf{D} \tilde{\mathbf{B}}}{3K_g} dS \mathbf{u}^e + \int_{S_q} \mathbf{N}_p J \rho_w^0 \bar{\mathbf{q}} \cdot \\ & \left[\frac{(1 - n^0) \mathbf{N}_p}{JK_g} - \frac{\mathbf{m}^T \mathbf{D} \mathbf{m} \mathbf{N}_p}{9K_g^2} - \frac{\mathbf{N}_p}{K_w} + \frac{\mathbf{m}^T \mathbf{D} \tilde{\mathbf{B}} \mathbf{u}^e \mathbf{N}_p}{3K_g K_w} + \right. \\ & \left. \frac{\mathbf{m}^T \mathbf{D} \mathbf{m} (\mathbf{N}_p \mathbf{p}_w^e - \rho_w^0) \mathbf{N}_p}{9K_g^2 K_w} + \frac{\mathbf{m}^T \tilde{\boldsymbol{\sigma}}^0 \mathbf{N}_p}{3K_g K_w} - \frac{(1 - n^0)}{K_w J} \cdot \right. \\ & \left. \left(\frac{\mathbf{N}_p \mathbf{p}_w^e}{K_g} - 1 \right) \right] dS \mathbf{p}_w^e - \int_{S_q} \frac{\mathbf{N}_p J n^T k \mathbf{g}}{g} - \mathbf{N}_p J \rho_w^0 \bar{\mathbf{q}} \cdot \\ & \left(1 + \frac{\mathbf{m}^T \mathbf{D} \mathbf{m} \rho_w^0}{9K_g^2} - \frac{\mathbf{m}^T \tilde{\boldsymbol{\sigma}}^0}{3K_g} - \frac{1 - n^0}{J} \right) dS = 0 \end{aligned} \quad (32)$$

式中： $\tilde{\mathbf{B}}$ 的表达式见文献[12]。

式（31）、（32）联立为储层应力渗流耦合的有限元方程，写成矩阵形式为

$$\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{M} & \mathbf{N} \end{bmatrix} \begin{Bmatrix} \mathbf{u}^e \\ \mathbf{p}_w^e \end{Bmatrix} + \begin{bmatrix} \mathbf{K} & \mathbf{L} \\ \mathbf{H} & \mathbf{S} \end{bmatrix} \frac{d}{dt} \begin{Bmatrix} \mathbf{u}^e \\ \mathbf{p}_w^e \end{Bmatrix} = \begin{Bmatrix} \mathbf{F} \\ \mathbf{V} \end{Bmatrix} \quad (33)$$

式中：

$$\mathbf{K} = \int_V (\mathbf{B}^T \mathbf{D} \mathbf{B} - \mathbf{B}^T \mathbf{N}_p \mathbf{p}_w^e \mathbf{m} \mathbf{m}^T \mathbf{B} + \mathbf{B}^T \mathbf{T} \mathbf{N} + \mathbf{G}^T \mathbf{S} \mathbf{N}) dV \quad (34)$$

$$\mathbf{L} = \int_V \left(\frac{\mathbf{B}^T \mathbf{D} \mathbf{m}}{3K_g} - \mathbf{B}^T \mathbf{m} \right) \mathbf{N}_p dV \quad (35)$$

$$\mathbf{F} = \frac{1}{dt} \left(\int_{S_\sigma} \mathbf{N}^T d\mathbf{t} dS + \int_V \mathbf{N}^T d\mathbf{f} dV \right) \quad (36)$$

$$\mathbf{M} = -J \rho_w^0 \int_{S_q} \frac{\mathbf{N}_p \bar{\mathbf{q}} \mathbf{m}^T \mathbf{D} \tilde{\mathbf{B}}}{3K_g} dS \quad (37)$$

$$\begin{aligned} \mathbf{N} = J \rho_w^0 \int_{S_q} \mathbf{N}_p \bar{\mathbf{q}} \left[\frac{(1 - n^0) \mathbf{N}_p}{JK_g} - \frac{\mathbf{m}^T \mathbf{D} \mathbf{m} \mathbf{N}_p}{9K_g^2} - \frac{\mathbf{N}_p}{K_w} + \right. \\ \left. \frac{\mathbf{m}^T \mathbf{D} \tilde{\mathbf{B}} \mathbf{u}^e \mathbf{N}_p}{3K_g K_w} + \frac{\mathbf{m}^T \mathbf{D} \mathbf{m} (\mathbf{N}_p \mathbf{p}_w^e - \rho_w^0) \mathbf{N}_p}{9K_g^2 K_w} + \right. \\ \left. \frac{\mathbf{m}^T \tilde{\boldsymbol{\sigma}}^0 \mathbf{N}_p}{3K_g K_w} - \frac{(1 - n^0)}{K_w J} \left(\frac{\mathbf{N}_p \mathbf{p}_w^e}{K_g} - 1 \right) \right] dS \end{aligned} \quad (38)$$

$$\begin{aligned} \mathbf{H} = J \rho_w^0 \int_V \mathbf{N}_p \left(1 + \frac{\mathbf{N}_p \mathbf{p}_w^e}{K_w} \right) \left[\mathbf{m}^T \mathbf{B} - \frac{\mathbf{m}^T \mathbf{D} \tilde{\mathbf{B}} \mathbf{u}^e \mathbf{m}^T \mathbf{B}}{3K_g} - \right. \\ \left. \frac{\mathbf{m}^T \mathbf{D} \mathbf{m} (\mathbf{N}_p \mathbf{p}_w^e - \rho_w^0) \mathbf{m}^T \mathbf{B}}{9K_g^2} - \frac{\mathbf{m}^T \tilde{\boldsymbol{\sigma}}^0 \mathbf{m}^T \mathbf{B}}{3K_g} - \frac{\mathbf{m}^T \mathbf{D} \mathbf{B}}{3K_g} \right] dV \end{aligned} \quad (39)$$

$$\begin{aligned} \mathbf{S} = \int_V \mathbf{N}_p \left[\frac{\rho_w^0 J}{K_w} - \frac{\rho_w^0 J}{K_w} \left(\frac{\mathbf{m}^T \mathbf{D} \tilde{\mathbf{B}} \mathbf{u}^e}{3K_g} + \right. \right. \\ \left. \left. \frac{\mathbf{m}^T \mathbf{D} \mathbf{m} (\mathbf{N}_p \mathbf{p}_w^e - \rho_w^0)}{9K_g^2} + \frac{\mathbf{m}^T \tilde{\boldsymbol{\sigma}}^0}{3K_g} \right) + \frac{\rho_w^0}{K_w} (1 - n^0) \cdot \right. \\ \left. \left(\frac{\mathbf{N}_p \mathbf{p}_w^e}{K_g} - 1 \right) - \frac{\mathbf{J} \mathbf{m}^T \mathbf{D} \mathbf{m}}{9K_g^2} + \frac{1 - n^0}{K_g} \right] \mathbf{N}_p dV \end{aligned} \quad (40)$$

$$\begin{aligned} \mathbf{V} = J \int_{S_q} \frac{\mathbf{N}_p n^T k \mathbf{g}}{g} - \mathbf{N}_p \rho_w^0 \bar{\mathbf{q}} \cdot \\ \left(1 + \frac{\mathbf{m}^T \mathbf{D} \mathbf{m} \rho_w^0}{9K_g^2} - \frac{\mathbf{m}^T \tilde{\boldsymbol{\sigma}}^0}{3K_g} - \frac{1 - n^0}{J} \right) dS \end{aligned} \quad (41)$$

有限元方程式（33）是由单元建立的，对其进行单元组装可得到整体有限元方程。

4 有限元方程的时域离散

时域上的离散可以看成一维有限元离散，时间单元内的值由节点值插值得到

$$\mathbf{u}^e = \mathbf{N}_n (\mathbf{u}^e)^n + \mathbf{N}_{n+1} (\mathbf{u}^e)^{n+1} \quad (42)$$

$$\mathbf{p}_w^e = \mathbf{N}_n (\mathbf{p}_w^e)^n + \mathbf{N}_{n+1} (\mathbf{p}_w^e)^{n+1} \quad (43)$$

式中：带上标 n 和 $n+1$ 的变量表示第 n 增量步和第 $n+1$ 增量步的节点值； $\mathbf{N}_n = 1 - \alpha$ ； $\mathbf{N}_{n+1} = \alpha$ ； $\alpha =$

$\frac{t-t_n}{\Delta t}$; t_n 为第 n 增量步的时刻, Δt 为第 n 步到第 $n+1$ 步的时间增量。

对式 (42) 和式 (43) 求导, 可得

$$\frac{d\mathbf{u}^e}{dt} = \frac{(\mathbf{u}^e)^{n+1} - (\mathbf{u}^e)^n}{\Delta t} \quad (44)$$

$$\frac{dp_w^e}{dt} = \frac{(p_w^e)^{n+1} - (p_w^e)^n}{\Delta t} \quad (45)$$

将式 (42)~(45) 代入式 (33), 整理可得

$$\begin{bmatrix} \mathbf{K} & \mathbf{L} \\ \mathbf{H} + \alpha \mathbf{M} \Delta t & \mathbf{S} + \alpha \mathbf{N} \Delta t \end{bmatrix} \begin{Bmatrix} \Delta(\mathbf{u}^e)^{n+1} \\ \Delta(p_w^e)^{n+1} \end{Bmatrix} = \begin{Bmatrix} \mathbf{F} \\ \mathbf{V} - \mathbf{M}(\mathbf{u}^e)^n - \mathbf{N}(p_w^e)^n \end{Bmatrix} \Delta t \quad (46)$$

式中:

$$\Delta(\mathbf{u}^e)^{n+1} = (\mathbf{u}^e)^{n+1} - (\mathbf{u}^e)^n, \Delta(p_w^e)^{n+1} = (p_w^e)^{n+1} - (p_w^e)^n \quad (47)$$

即为储层流固耦合瞬态非线性增量有限元方程, 采用牛顿法求解, 其系数矩阵采用非对称矩阵存储。初始时刻单元节点位移 $(\mathbf{u}^e)^0$ 和单元节点孔隙压力 $(p_w^e)^0$ 为已知条件, 通过反复迭代可以求出第 $n+1$ 增量步的单元节点位移增量 $\Delta(\mathbf{u}^e)^{n+1}$ 和单元节点孔隙压力增量 $\Delta(p_w^e)^{n+1}$, 从而求出第 $n+1$ 增量步时刻的单元节点位移 $(\mathbf{u}^e)^{n+1}$ 和单元节点孔隙压力 $(p_w^e)^{n+1}$, 进一步计算得到相应的位移、应力和孔隙压力分布。

5 结 论

(1) 列出了固体骨架的控制方程, 根据有效应力原理考虑了流体孔隙压力对固体骨架有效应力的影响, 基于大变形增量理论, 得到了固体骨架有限元方程。

(2) 列出了流体的连续性方程, 考虑了固体骨架有效应力和流体孔隙压力共同对多孔介质孔隙度的影响, 采用 Galerkin 法, 得到了流体有限元方程。

(3) 对系统有限元方程进行了时域离散, 得到了以单元节点位移和单元节点孔隙压力为未知量的流固耦合瞬态非线性增量有限元方程。

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